EXPANSIONS IN TERMS OF SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS

SECOND PAPER: MULTIPLE BIRKHOFF SERIES*

BY

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1. Setting of the problem

Consider the system of expressions

$$L_i(z) \equiv \frac{\partial^{n_i} z}{\partial x_i^{n_i}} + P_{i2}(x_i)\frac{\partial^{n_i-2} z}{\partial x_i^{n_i-2}} + \cdots + P_{in_i}(x_i)z \qquad (i = 1, 2, \ldots, \varkappa)$$

and their adjoints

$$M_i(z) \equiv (-1)^{n_i}\frac{\partial^{n_i} z}{\partial x_i^{n_i}} + (-1)^{n_i-2}\frac{\partial^{n_i-2}}{\partial x_i^{n_i-2}}[P_{i2}(x_i)z] + \cdots + P_{in_i}(x_i)z \qquad (i = 1, 2, \ldots, \varkappa),$$

in which the coefficients $P_{ij}(x_i)$ $(i = 1, 2, \ldots, \varkappa;\ j = 1, 2, \ldots, n_i)$ are functions of the real variables x_i on closed intervals (a_i, b_i), which are continuous with their derivatives of all orders.

With the partial differential equation

$$(1) \qquad \sum_{i=1}^{\varkappa} L_i(u) + \lambda u = 0$$

and the boundary conditions

$$(2) \qquad T_{ij}(u) = 0 \qquad (i = 1, 2, \ldots, \varkappa;\ j = 1, 2, \ldots, n_i)$$

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we associate the adjoint equation

$$\sum_{i=1}^{\varkappa} M_i(v) + \lambda v = 0 \tag{3}$$

and certain adjoint boundary conditions

$$U_{ij}(v) = 0 \qquad (i = 1, 2, \ldots, \varkappa; j = 1, 2, \ldots, n_i); \tag{4}$$

where

$$\begin{aligned} T_{ij}(u) \equiv & \left[\alpha_{j0}^{(i)} u(x_1, x_2, \ldots, x_\varkappa) + \sum_{h=1}^{n_i-1} \alpha_{jh}^{(i)} \frac{\partial^h u}{\partial x_i^h}\right]_{x_i=a} \\ & + \left[\beta_{j0}^{(i)} u(x_1, x_2, \ldots, x_\varkappa) + \sum_{h=1}^{n_i-1} \beta_{jh}^{(i)} \frac{\partial^h u}{\partial x_i^h}\right]_{x_i=b_i} \end{aligned} \tag{5}$$

and $U_{ij}(v)$ is to be defined later.

If we make the substitutions

$$u \equiv \prod_{i=1}^{\varkappa} u_i(x_i), \qquad v \equiv \prod_{i=1}^{\varkappa} v_i(x_i), \tag{6}$$

and omit the trivial solutions, equation (1) reduces as in my first paper* to the system of ordinary differential equations

$$L_i(u_i) + \mu_i u_i = 0 \qquad (i = 1, 2, \ldots, \varkappa), \tag{7}$$

where $\sum \mu_i = \lambda$ and equation (3), by a suitable choice of μ's, to the adjoint system. Likewise the boundary conditions (2) upon being divided through by appropriate factors take the form

$$\begin{aligned} W_{ij}(u_i) \equiv & \alpha_{j0}^{(i)} u_i(a_i) + \beta_{j0}^{(i)} u_i(b_i) \\ & + \sum_{h=1}^{n_i-1} \left[\alpha_{jh}^{(i)} \frac{d^h u_i(a_i)}{dx_i^h} + \beta_{jh}^{(i)} \frac{d^h u_i(b_i)}{dx_i^h}\right] = 0 \end{aligned} \tag{8}$$

$$(i = 1, 2, \ldots, \varkappa; j = 1, 2, \ldots, n_i).$$

* These Transactions, vol. 25 (1923), pp. 123–134.

For a particular i we then have a Birkhoff system. We shall restrict the α's and β's so that the boundary conditions are what he calls *regular*.* To each set W_{ij} corresponds one of his adjoint sets of boundary conditions,

$$V_{ij}(v_i) = 0 \qquad (i = 1, 2, \ldots, \varkappa;\ j = 1, 2, \ldots, n_i). \tag{9}$$

We now define $U_{ij}(v)$ to bear the same relation to $V_{ij}(v_i)$ that $T_{ij}(u)$ bears to $W_{ij}(u_i)$. Thus we ensure that for each i the parameter values $\mu_i^{(1)}, \mu_i^{(2)}, \ldots$ will be in general simple, and the principal solutions unique except for constant factors. For two distinct principal parameter values $\mu_i^{(m)}, \mu_i^{(n)}$ the orthogonal relation

$$\int_{a_i}^{b} u_i^{(m)}(x_i)\, v_i^{(n)}(x_i)\, dx_i = 0 \tag{10}$$

holds for the corresponding principal solutions.

2. The formal expansion

The object of this paper is to develop a more or less arbitrary function $f(x_1, x_2, \ldots, x_\varkappa)$ in the form

$$f = \sum^{\infty}\sum^{\infty}\cdots\sum^{\infty}_{h_1, h_2, \ldots, h_\varkappa = -\infty} C_{h_1, h_2, \ldots, h_\varkappa}\, U_{h_1, h_2, \ldots, h_\varkappa}(x_1, x_2, \ldots, x_\varkappa) \tag{11}$$

where $U_{h_1, h_2, \ldots, h_\varkappa} \equiv \prod_{i=1}^{\varkappa} u_i^{(h_i)}(x_i)$ is the solution of (1), (2) corresponding to the characteristic values $\mu_i^{(h_i)}$ $(i = 1, 2, \ldots, \varkappa)$ for the systems (7), (8). By multiplying (11) by the solution of (3), (4) for the same values of μ_i, integrating and using (6), (10) we obtain in general

$$C_{h_1, h_2, \ldots, h_\varkappa} = \frac{\displaystyle\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_\varkappa}^{b_\varkappa} f \prod_{i=1}^{\varkappa} v_i^{(h_i)}(x_i)\, dx_i}{\displaystyle\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_\varkappa}^{b_\varkappa} \prod_{i=1}^{\varkappa} u_i^{(h_i)}(x_i)\, v_i^{(h_i)}(x_i)\, dx_i}. \tag{12}$$

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* These Transactions, vol. 9 (1908), pp. 382–3.

In case the parameter values are not all simple or the Green's functions for the systems (7), (8) have poles not all of the first order, we replace the corresponding term of (11) by

$$(13) \qquad \int_{a_\varkappa}^{b_\varkappa}\int_{a_{\varkappa-1}}^{b_{\varkappa-1}}\cdots\int_{a_1}^{b_1} f(s_1, s_2, \ldots, s_\varkappa)\prod_{i=1}^{\varkappa} R_i^{(h_i)}(x_i, s_i)\, ds_i$$

where $R_i^{(h_i)}$ is the residue of the Green's function $G_i(x_i, s_i; \mu_i)$ $(i = 1, 2, \ldots, \varkappa)$ for the characteristic value $\mu_i^{(h_i)}$. Such an expansion may well be called a *multiple Birkhoff series.*

3. Convergence of the series

The value of the series is found as in my first paper* by taking the limit of contour integrals. The residue for a function of $\varkappa$ complex variables is defined as before. One makes the transformation $\mu_i = \varrho_i^{n_i}$ analogous to that of Birkhoff, $\lambda = \varrho^n$. Then one evaluates the limit of

$$\frac{1}{(2\pi\sqrt{-1})^\varkappa}\int_{\Gamma_\varkappa}\cdots\int_{\Gamma_1}\int_{a_\varkappa}^{b_\varkappa}\cdots\int_{a_1}^{b_1} f(s_1, \ldots, s_\varkappa)\prod_{i=1}^{\varkappa} G_i(x_i, s_i; \mu_i)\, ds_i\, d\mu_i$$

quite readily on account of the fact that the integrals separate in pairs. We may therefore state the

THEOREM. *Let $f(x_1, x_2, \ldots, x_\varkappa)$ be made up of a finite number of pieces in the region $a_i \leq x_i \leq b_i$ $(i = 1, 2, \ldots, \varkappa)$, each real, continuous, and possessing continuous partial derivatives. The multiple Birkhoff expansion connected with the partial differential equation* (1) *and the regular boundary conditions* (2) *converges to the mean value $\frac{1}{2^\varkappa}\sum f(x_1 \pm 0, x_2 \pm 0, \ldots, x_\varkappa \pm 0)$ at any interior point of the region. In any closed subregion in which f is continuous and possesses continuous partial derivatives the series converges uniformly to f.*

4. Applications to physical problems

Several of the most important differential equations of physics give rise to special cases of equation (1). For instance the wave equation $\partial^2\varphi/\partial t^2 = c\nabla^2\varphi$

* Loc. cit.

has a large class of solutions of the form $\varphi = T(t)\, u(x, y, z)$. These depend on solutions of $T'' + \lambda t = 0$ and $c\nabla^2 u + \lambda u = 0$. The boundary conditions are usually regular.

The flow of heat, vibrations of a drumhead, and determination of potential also lead to forms of equation (1). Although the boundary conditions (2) for three dimensions seem to restrict us to values on a parallelepiped, it is important to notice that any transformation leading to generalized coördinates which changes an equation to another form included in (1) will allow us to treat more general boundary conditions involving values on pairs of mutually orthogonal surfaces. A similar remark applies to the system (7).

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